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ON THE REMAINDER OF GAUSSIAN QUADRATURE FORMULAS FOR BERNSTEIN-SZEGÖ WEIGHT FUNCTIONS

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ABSTRACT. We give an explicit expression for the kernel of the error functional for Gaussian quadrature formulas with respect to weight functions of Bernstein-Szegö type, i.e., weight functions of the form $(1-x)^{\alpha}(1+x)^{\beta}/\rho(x)$, $x \in (-1, 1)$, where $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ and ρ is a polynomial of arbitrary degree which is positive on $[-1, 1]$. With the help of this result the norm of the error functional can easily be calculated explicitly for a wide subclass of these weight functions.

1. INTRODUCTION AND NOTATION

We consider Gaussian quadrature formulas with respect to a nonnegative weight function w on the interval $[-1, 1]$,

$$\int_{-1}^{1} f(x)w(x)\,dx = \sum_{j=1}^{n} \lambda_j f(x_j) + R_n(f, w), \tag{1.1}$$

where $x_j = x_{j,n}$ are the zeros of the nth-degree monic orthogonal polynomial $P_n(\cdot, w)$ and $\lambda_j = \lambda_{j,n}$ are the corresponding Christoffel numbers. If f is analytic in a domain D which contains in its interior the interval $[-1, 1]$ and a contour Γ surrounding $[-1, 1]$, the remainder term can be represented as a contour integral (see, e.g., [3])

$$R_n(f, w) = \frac{1}{2\pi i}\int_{\Gamma} K_n(z, w) f(z)\,dz, \tag{1.2}$$

where the kernel $K_n(\cdot, w)$ is given by

$$K_n(z, w) = R_n\left(\frac{1}{z-\cdot}, w\right) \tag{1.3}$$

or, alternatively, by

$$K_n(z, w) = \frac{Q_n(z, w)}{P_n(z, w)}, \tag{1.4}$$

where $Q_n(\cdot, w)$ is the nth function of the second kind, i.e.,

$$Q_n(z, w) = \int_{-1}^{1} \frac{P_n(x, w)}{z-x} w(x)\,dx \quad \text{for } z \in \mathbf{C}\backslash[-1, 1]. \tag{1.5}$$

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Let us note that by (1.3), $K_n(\cdot, w)$ has the following series expansion:

$$K_n(z, w) = \sum_{k=2n}^{\infty} \frac{R_n(x^k, w)}{z^{k+1}} \quad \text{for } |z| > 1. \tag{1.6}$$

From (1.2) the following well-known estimate of the remainder, based on contour integration, follows:

$$|R_n(f, w)| \le \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma} |K_n(z, w)| \max_{z\in\Gamma} |f(z)|, \tag{1.7}$$

where $l(\Gamma)$ denotes the length of Γ.

Another useful method to estimate the remainder for a function analytic in $C_r = \{z \in \mathbf{C}: |z| < r\}$, $r > 1$, has been suggested by Hämmerlin [4], namely: For a function $f(z) = \sum_{k=0}^{\infty} a_k(f) z^k$ analytic in C_r define

$$|f|_r := \sup\{|a_k(f)| r^k : k \in \mathbf{N}_0 \text{ and } R_n(x^k, w) \ne 0\}.$$

Then, $|\cdot|_r$ in the space

$$\mathfrak{X}_r := \{f : f \text{ analytic in } C_r \text{ and } |f|_r < \infty\}$$

is a seminorm. The error functional $R_n(f, w)$ is continuous in $(\mathfrak{X}_r, |\cdot|_r)$, and we have

$$|R_n(f, w)| \le \|R_n\| \, |f|_r,$$

where $\|R_n\|$ can be estimated by

$$\|R_n\| \le \sum_{k=2n}^{\infty} \frac{|R_n(x^k, w)|}{r^k}. \tag{1.8}$$

Equality holds (put $f(z) = 1/(r - z)$, resp. $1/(r + z)$) if for all $k \ge 2n$ the condition

$$R_n(x^k, w) \ge 0, \quad \text{resp. } (-1)^k R_n(x^k, w) \ge 0, \tag{1.9}$$

is fulfilled. Since by [3, Theorem 2.1], proved in [2], and the proof of Theorem 3.1 in [3], the condition

$$w(x)/w(-x) \text{ nondecreasing on } (-1, 1), \tag{1.10}$$

resp.

$$w(x)/w(-x) \text{ nonincreasing on } (-1, 1), \tag{1.11}$$

implies that condition (1.9) holds for all $k \in \mathbf{N}_0$, it follows by (1.6) (see [3, Theorem 3.1]) that

$$\max_{|z|=r} |K_n(z, w)| = \begin{cases} K_n(r, w) & \text{if } w \text{ satisfies (1.10)}, \\ |K_n(-r, w)| & \text{if } w \text{ satisfies (1.11)}, \end{cases}$$

and

$$\|R_n\| = \begin{cases} rK_n(r, w) & \text{if } w \text{ satisfies (1.10)}, \\ -rK_n(-r, w) & \text{if } w \text{ satisfies (1.11)}. \end{cases}$$

Thus, we see that for the estimation of the remainder it is very desirable to have an explicit expression for the kernel $K_n(z, w)$.

Very recently, Notaris [8] computed $\|R_n\|$ explicitly for weight functions of the form

$$w(x) = (1-x)^\alpha(1+x)^\beta/\rho_2(x) \quad \text{for } x \in (-1, 1), \tag{1.12}$$

where $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ and ρ_2 is a polynomial of degree at most two which is positive on $[-1, 1]$ and satisfies condition (1.10) or (1.11). For the special case when ρ_2 is a polynomial of degree one or a particular even polynomial of degree two, this has been done before by Akrivis [1] (see also Kumar [5, 6]). Let us also mention that a detailed study of the kernel function for the four Chebyshev weight functions, i.e., $\rho_2(x) \equiv 1$ in (1.12), can be found in Gautschi and Varga [3]. In this note we derive an explicit expression for the kernel $K_n(z, w)(\|R_n\|)$ for all Bernstein-Szegö weight functions w (which satisfy condition (1.10) or (1.11)), where a weight function is called a Bernstein-Szegö weight function if it is of the form

$$\pi w_{\alpha,\beta}(x, \rho_m) = (1-x)^\alpha(1+x)^\beta/\rho_m(x) \quad \text{for } x \in (-1, 1), \tag{1.13}$$

with $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$ and ρ_m a polynomial of degree m, m arbitrary, which is positive on $[-1, 1]$.

2. Main result

First let us recall the well-known fact that the so-called Joukowski transformation

$$y = \tfrac{1}{2}(z + z^{-1}) \tag{2.1}$$

maps $\{z \in \mathbf{C}: |z| < 1\}\backslash\{0\}$ $(\{z \in \mathbf{C}: |z| > 1\})$ one-to-one onto $\mathbf{C}\backslash[-1, 1]$ and that the inverse transformation is given by

$$z = y \underset{(+)}{-} \sqrt{y^2 - 1}, \tag{2.2}$$

where that branch of $\sqrt{\ }$ is chosen for which $\sqrt{y^2-1} > 0$ for $y \in (1, \infty)$. Note that the transformation (2.1) maps the circumference $|z| = 1$ onto the interval $[-1, 1]$.

The following version of the Fejér-Riesz Theorem on the representation of positive trigonometric polynomials (compare Theorems 1.2.1 and 1.2.2 in [10]) will be needed.

Lemma. *Let ρ_m be a real positive polynomial on $[-1, 1]$ of exact degree m. Then there exists a unique real polynomial*

$$g_m(z) = \prod_{\nu=1}^{m}(z - z_\nu) \quad \textit{with } 0 < |z_\nu| < 1 \textit{ for } \nu = 1, \dots, m \tag{2.3}$$

such that

$$\rho_m(\cos\varphi) = c|g_m(e^{i\varphi})|^2 \quad \textit{for } \varphi \in [0, 2\pi], \tag{2.4}$$

where $c \in \mathbf{R}^+$.

Proof. Let $\rho_m(x) = \tilde{c}\prod_{\nu=1}^m(\alpha_\nu - x)$, where $\tilde{c} \in \mathbf{R}$ and the α_ν's are either in $\mathbf{R}\backslash[-1, 1]$ or appear in pairs of complex conjugate numbers. Hence, if we set

$$z_\nu = \alpha_\nu - \sqrt{\alpha_\nu^2 - 1} \quad \text{for } \nu = 1, \dots, m, \tag{2.5}$$

then

$$0<|z_\nu|<1 \quad \text{and} \quad \alpha_\nu = \tfrac{1}{2}(z_\nu + z_\nu^{-1}) \quad \text{for } \nu = 1, \dots, m.$$

Thus,

$$\rho_m(\cos\varphi) = \tilde{c}\prod_{\nu=1}^{m}\left(\frac{1+z_\nu^2}{2z_\nu} - \cos\varphi\right) = c\prod_{\nu=1}^{m}|e^{i\varphi} - z_\nu|^2,$$

since

$$(e^{i\varphi} - z_\nu)(e^{-i\varphi} - z_\nu) = 2z_\nu\left(\frac{1+z_\nu^2}{2z_\nu} - \cos\varphi\right)$$

and the z_ν's are real or appear in pairs of complex conjugate numbers.

Now the uniqueness remains to be shown. Suppose that

$$\rho_m(\cos\varphi) = d\prod_{\nu=1}^{m}|e^{i\varphi} - v_\nu|^2,$$

where $d \in \mathbf{R}^+$, $v_\nu \in \{z \in \mathbf{C}\colon |z| < 1\}\backslash\{0\}$ for $\nu = 1, \dots, m$, and the v_ν's are real or complex conjugate. Then it follows as above that $(v_\nu + v_\nu^{-1})/2$, $\nu = 1, \dots, m$, is a zero of $\rho_m(\cos\varphi)$ and thus, since $0 < |v_\nu| < 1$ and since the Joukowski transformation is one-to-one, the uniqueness follows. □

Let us note that other representations of ρ_m of the form (2.4), but with g_m having $m-l$, resp. l, $l \in \{1, \dots, m\}$, zeros inside, resp. outside, of the unit disk, can be obtained by replacing (2.5) by

$$z_{\nu_j} = \alpha_{\nu_j} + \sqrt{\alpha_{\nu_j}^2 - 1} \quad \text{for } j = 1, \dots, l, \tag{2.6}$$

where $\{\nu_1, \dots, \nu_l\}$ is an arbitrary subset of $\{1, \dots, m\}$, and

$$z_\nu = \alpha_\nu - \sqrt{\alpha_\nu^2 - 1} \quad \text{for } \nu \in \{1, \dots, m\}\backslash\{\nu_1, \dots, \nu_l\}.$$

Now let us set

$$\pi w(x, \rho_m) = 1/(\sqrt{1-x^2}\,\rho_m(x)) \quad \text{for } x \in (-1, 1)$$

and let g_m be the unique polynomial from the above lemma. Then it follows by well-known results of Bernstein and Szegö (see, e.g., [10, p. 31] and set $h_m(z) = \sqrt{c\pi}z^m g_m(\frac{1}{z})$ there) that, with $z = e^{i\varphi}$ and $x = \cos\varphi$,

$$\begin{aligned}
2^{n-1}P_n(x, w(\cdot, \rho_m)) &= \sum_{j=0}^{m} a_j T_{n-j}(x)\\
&= \mathrm{Re}\{z^{n-m}g_m(z)\} \quad \text{for } 2n > m,\\
2^{n-1}P_{n-1}(x, (1-x^2)w(\cdot, \rho_m)) &= \sum_{j=0}^{m} a_j U_{n-1-j}(x)\\
&= \mathrm{Im}\{z^{n-m}g_m(z)\}/\sin\varphi \quad \text{for } 2n > m,\\
2^n P_n(x, (1+x)w(\cdot, \rho_m)) &= \sum_{j=0}^{m} a_j \frac{T_{n+1-j}(x) + T_{n-j}(x)}{x+1}\\
&= \mathrm{Re}\{z^{n-m+1/2}g_m(z)\}/\cos(\varphi/2) \quad \text{for } 2n+1 > m,\\
2^n P_n(x, (1-x)w(\cdot, \rho_m)) &= \sum_{j=0}^{m} a_j \frac{T_{n+1-j}(x) - T_{n-j}(x)}{x-1}\\
&= \mathrm{Im}\{z^{n-m+1/2}g_m(z)\}/\sin(\varphi/2) \quad \text{for } 2n+1 > m,
\end{aligned} \tag{2.7}$$

where $g_m(z) = \sum_{j=0}^{m} a_j z^{m-j}$ and T_j, resp. U_j, denotes the Chebyshev polynomial of degree j of the first, resp. second, kind on $[-1, 1]$.

We mention in passing that if g_m in (2.7) is replaced by a polynomial $\tilde{g}_m$ which also satisfies (2.4) but does not have all zeros in the open unit disk, then the polynomials on the right-hand side in (2.7) are not orthogonal with respect to $w_{\alpha,\beta}(\cdot, \rho_m)$, $\alpha, \beta \in \{-\frac{1}{2}, \frac{1}{2}\}$. In fact (see [9, Corollary 5], corresponding results hold also for $\alpha = -\beta = \pm\frac{1}{2}$), they are orthogonal with respect to a functional Ψ of the form

$$\Psi(p) = \int_{-1}^{+1} p(x) w_{\alpha,\beta}(x, \rho_m)\, dx + L(p) \quad \text{for } p \in \mathbf{P},$$

where L is a functional given by

$$L(p) = \sum_{j=1}^{l^*} \sum_{\kappa=1}^{l_j} \mu_{\kappa,j} p^{(\kappa-1)}(\alpha_{\nu_j}),$$

and the α_{ν_j}'s are those zeros of ρ_m which correspond to the zeros of $\tilde{g}_m$ lying outside of the unit disk by (2.6), l_j is the multiplicity of the zero α_{ν_j}, and the $\mu_{k,j}$'s are certain real numbers.

We now give the announced explicit expression for the kernel function $K_n(z, w_{\alpha,\beta})$, $|\alpha| = |\beta| = \frac{1}{2}$.

Theorem. *Let ρ_m be given by* (2.4). *Then we have for $y \in \mathbf{C}\backslash[-1, 1]$, on writing $y = \frac{1}{2}(z + z^{-1})$ with $|z| < 1$, i.e., $z = y - \sqrt{y^2 - 1}$, that*

$$cK_n(y, w(\cdot, \rho_m)) = \frac{4z^{2n+1}}{(1 - z^2) g_m^*(z)\{z^{2n-m} g_m(z) + g_m^*(z)\}} \quad \textit{for } 2n > m,$$

$$cK_n(y, (1 \overset{+}{\underset{(-)}{}} x) w(\cdot, \rho_m)) = \frac{2z^{2n+1}(z \overset{+}{\underset{(-)}{}} 1)}{(1 \overset{-}{\underset{(+)}{}} z) g_m^*(z)\{z^{2n+1-m} g_m(z) \overset{+}{\underset{(-)}{}} g_m^*(z)\}} \quad \textit{for } 2n + 1 > m,$$

$$cK_n(y, (1 - x^2) w(\cdot, \rho_m)) = \frac{z^{2n+1}(z^2 - 1)}{g_m^*(z)\{z^{2n+2-m} g_m(z) - g_m^*(z)\}} \quad \textit{for } 2n + 2 > m,$$

where $g_m^*(z) = z^m g_m(\frac{1}{z})$.

Proof. Let R and S be monic polynomials of degree at most two such that

$$R(y)S(y) = y^2 - 1,$$

and let us put for abbreviation

$$P_n(x) := P_n(x, Rw(\cdot, \rho_m)) \quad \text{and} \quad \widetilde{P}_n(x) := P_n(x, Sw(\cdot, \rho_m)).$$

Using the simple fact that for $k \in \mathbf{Z}$ and $\varphi \in [0, 2\pi]$

$$[\mathrm{Re}\{e^{ik\varphi} g_m(e^{i\varphi})\}]^2 + [\mathrm{Im}\{e^{ik\varphi} g_m(e^{i\varphi})\}]^2 = |g_m(e^{i\varphi})|^2, \tag{2.8}$$

we get, using (2.7) and (2.4), that with $l = n + \partial R - 1$, where ∂R denotes the exact degree of R,

$$RP_n^2 - S\widetilde{P}_l^2 = k_n \rho_m, \tag{2.9}$$

where

$$k_n = \begin{cases} 2^{-2n+2}/c & \text{for } R(x) \equiv 1\,, \\ -2^{-2n}/c & \text{for } R(x) \equiv x^2 - 1\,, \\ \overset{+}{(-)} 2^{-2n+1}/c & \text{for } R(x) \equiv x + 1 \ (x-1)\,. \end{cases}$$

Furthermore, it follows from Theorem 3(a) of our paper [9] that for $2n \geq m + 1 - \partial R$

$$RP_n^2 - S(YP_n + \rho_m P_{n-1}^{(1)})^2 = d_n \rho_m\,, \tag{2.10}$$

where $P_{n-1}^{(1)}$ denotes the associated polynomial of P_n, i.e.,

$$P_{n-1}^{(1)}(y) = \int_{-1}^{1} \frac{P_n(y) - P_n(x)}{y - x} R(x) w(x\,,\, \rho_m)\, dx\,,$$

and $Y \in \mathbf{P}_\mu$, $\mu = \max\{m-1\,,\, \partial R - 1\}$, is uniquely determined by the conditions that at each zero α_ν of $\rho_m(x) = \tilde{c} \prod_{\nu=1}^{m^*} (\alpha_\nu - x)^{m_\nu}$, where $\tilde{c} \in \mathbf{R}$ and m_ν is the multiplicity of the zero α_ν,

$$Y^{(j)}(\alpha_\nu) = (R/\sqrt{y^2-1})^{(j)}(\alpha_\nu) \quad \text{for } j = 0\,, \dots\,, m_\nu - 1\,, \tag{2.11}$$

and that for $y \to \infty$

$$\frac{(R/\sqrt{y^2-1} - Y)(y)}{\rho_m(y)} = O(y^{-1})\,; \tag{2.12}$$

furthermore,

$$d_n = 2 \int_{-1}^{+1} P_n^2(x) R(x) w(x\,,\, \rho_m)\, dx\,. \tag{2.13}$$

(We note that in the definition of $1/h$ in [9, p. 461] $(-1)^{l-k}/\sqrt{-H}$ is to be replaced by $(-1)^{l-k}/\pi\sqrt{-H}$.) It now follows from [10, (2.6.5)] that the leading coefficient of the orthonormal polynomial of degree n with respect to $Rw(\cdot\,,\, \rho_m)$ is equal to $\sqrt{2/k_n}$ for $2n \geq m+1-\partial R$, which implies that $d_n = k_n$ and thus, in view of (2.9) and (2.10),

$$\pm \widetilde{P}_l = YP_n + \rho_m P_{n-1}^{(1)} \quad \text{for } 2n \geq m + 1 - \partial R\,. \tag{2.14}$$

For a function f defined on $\mathbf{C}\backslash[-1\,,\, 1]$ and for $x \in (-1\,,\, 1)$ we write, provided the limits involved exist,

$$f^{\overset{+}{(-)}}(x) := \lim_{\substack{z \to x \\ z \in \mathbf{C}^{\overset{+}{(-)}}}} f(z)\,,$$

where $\mathbf{C}^{\overset{+}{(-)}} := \{z \in \mathbf{C}\colon \operatorname{Im} z \overset{>}{(<)} 0\}$. Observing that by (2.11) and (2.12)

$$\Phi(y) := \frac{(R/\sqrt{y^2-1} - Y)(y)}{\rho_m(y)} \tag{2.15}$$

is analytic on $\mathbf{C}\backslash[-1\,,\, 1]$ and vanishes at infinity, and that the boundary values $\Phi^{\pm}(x)$, $x \in (-1\,,\, 1)$, from the upper (lower) half plane satisfy the relation

$$\Phi^+(x) - \Phi^-(x) = \frac{2}{i} \frac{R(x)}{\rho_m(x)\sqrt{1-x^2}} \quad \text{for } x \in (-1\,,\, 1)\,,$$

where we have used the fact that

$$(\sqrt{y^2-1})^+(x) = i\sqrt{1-x^2} = -(\sqrt{y^2-1})^-(x),$$

we get by the Sochozki-Plemelj formula (see, e.g., [7])

$$\begin{aligned}\Phi(y) &= \frac{1}{\pi}\int_{-1}^{+1}\frac{1}{y-x}\frac{R(x)}{\rho_m(x)\sqrt{1-x^2}}\,dx \\ &= Q_0(y\,,\,Rw(\cdot\,,\,\rho_m)) \quad \text{for } y\in\mathbf{C}\backslash[-1\,,\,1].\end{aligned} \tag{2.16}$$

Recalling the well-known fact (see, e.g., [10, §3.5]) that for sufficiently large $|y|$

$$\frac{P_{n-1}^{(1)}(y)}{P_n(y)} = Q_0(y\,,\,Rw(\cdot\,,\,\rho_m)) + O(y^{-(2n+1)}),$$

we get, using (2.16) and (2.15), that

$$(YP_n + \rho_m P_{n-1}^{(1)})(y) = P_n(y)R(y)/\sqrt{y^2-1} + O(y^{-(n+1)+m})$$

and thus, since for $2n \ge m+1-\partial R$

$$\lim_{y\to\infty} y^{-(n+\partial R-1)}\{P_n(y)R(y)/\sqrt{y^2-1} + O(y^{-(n+1)+m})\} = 1,$$

the polynomial $YP_n + \rho_m P_{n-1}^{(1)}$, which by (2.14) is of exact degree $n+\partial R-1$, has leading coefficient one. Hence, the plus sign holds in (2.14). Thus, the nth function of the second kind is of the form

$$\begin{aligned}Q_n(y\,,\,Rw(\cdot\,,\,\rho_m)) &= -P_{n-1}^{(1)}(y) + P_n(y)Q_0(y) \\ &= \frac{\sqrt{R(y)}P_n(y) - \sqrt{S(y)}\widetilde{P}_l(y)}{\sqrt{S(y)}\rho_m(y)} = \frac{k_n}{(\sqrt{y^2-1}P_n + S\widetilde{P}_l)(y)},\end{aligned} \tag{2.17}$$

where the second equality follows with the help of (2.16), (2.15), and (2.14), and the third equality with the help of (2.9). Now the following equalities hold on the circumference $|z|=1$:

$$2^n P_n\left(\frac{1}{2}(z+z^{-1})\,,\,w(\cdot\,,\,\rho_m)\right) = z^{-n}(z^{2n-m}g_m(z) + g_m^*(z)),$$

$$\begin{aligned}&2^{n-1}P_{n-1}\left(\frac{1}{2}(z+z^{-1})\,,\,(1-x^2)w(\cdot\,,\,\rho_m)\right) \\ &\quad = \frac{z^{-n}(z^{2n-m}g_m(z) - g_m^*(z))}{z-z^{-1}}, \\ &2^n P_n\left(\frac{1}{2}(z+z^{-1})\,,\,(1 \overset{+}{\underset{(-)}{}} x)w(\cdot\,,\,\rho_m)\right) \\ &\quad = \frac{z^{-n}(z^{2n+1-m}g_m(z) \overset{+}{\underset{(-)}{}} g_m^*(z))}{z \overset{+}{\underset{(-)}{}} 1}.\end{aligned} \tag{2.18}$$

Since all functions appearing in (2.18) are analytic in the domain $\mathbf{C}\backslash\{0\}$, it follows that in (2.18) equality holds also on $\mathbf{C}\backslash\{0\}$. Hence, we get from (2.17) and (2.18) that for $y \in \mathbf{C} \backslash [-1, 1]$, on writing $y = \frac{1}{2}(z + z^{-1})$ with $|z| < 1$,

$$\begin{aligned} Q_n(y, w(\cdot, \rho_m)) &= \frac{2^{-n+2}}{c} \frac{z^{n+1}}{(1-z^2) g_m^*(z)}, \\ Q_n(y, (1-x^2) w(\cdot, \rho_m)) &= \frac{2^{-n}}{c} \frac{z^{n+1}}{g_m^*(z)}, \qquad (2.19) \\ Q_n(y, (1 \underset{(+)}{-} x) w(\cdot, \rho_m)) &= \frac{2^{-n+1}}{c} \frac{z^{n+1}}{(1 \underset{(-)}{+} z) g_m^*(z)}, \end{aligned}$$

where we have used the fact that $\sqrt{y^2 - 1} = (z^{-1} - z)/2$. Relation (2.19) in conjunction with (2.18) and (1.4) gives the assertion. □

In the remark below we state sufficient conditions on the weight function $w_{\alpha,\beta}(x, \rho_m)$, defined in (1.13), such that (1.10), resp. (1.11), is fulfilled. Since the product $w_1(x) w_2(x)$ of two weight functions w_1, w_2 satisfies condition (1.10), resp. (1.11) if w_1 and w_2 satisfy (1.10), resp. (1.11), we consider the behavior of $w(x, \rho_m)/w(-x, \rho_m)$ for $m \in \{1, 2\}$ only.

Remark. The ratio $w(x, \rho)/w(-x, \rho)$ is nondecreasing (nonincreasing) on $(-1, 1)$ if

$$\rho(x) = \begin{cases} \underset{(-)}{+}(\alpha - x), & \alpha \in (1, \infty) \ ((-\infty, -1)), \\ (\alpha - x)(x - \beta), & \alpha \in (1, \infty), \ \beta \in (-\infty, -1), \text{ and } -\beta \underset{(\le)}{\ge} \alpha, \\ (x-\alpha)^2 + \beta^2, & \alpha \in \mathbf{R}^{\overset{+}{(-)}}, \ \beta \in \mathbf{R}, \text{ and } \alpha^2 + \beta^2 \ge 1, \end{cases}$$

where the expressions in parentheses refer to the case of nonincreasing ratio.

Setting in the preceding theorem

$$g_1(z) = z + \tilde{a}, \quad \tilde{a} \in (-1, 1), \quad \text{i.e., } |g_1(e^{i\varphi})|^2 = 1 + \tilde{a}^2 + 2\tilde{a}x,$$

resp. for $b > 0$

$$g_2(z) = z^2 + (1 + 2b)^{-1}, \quad \text{i.e., } |g_2(e^{i\varphi})|^2 = 4(b^2 + (1+2b)x^2)/(2b+1)^2,$$

where $x = \cos\varphi$, we obtain the results of Kumar [5, 6] concerning the functions of the second kind, and the results of Akrivis [1] on the norm of the error functional $R_n(\cdot, w_{\alpha,\beta}(\cdot, |g_j(e^{i\varphi})|^2))$, $j = 1, 2$. If we put

$$g_2(z) = z^2 + \frac{2\delta}{\beta} z + \left(1 - \frac{2\alpha}{\beta}\right)$$

with $0 < \alpha < \beta$, $\beta \neq 2\alpha$, and $|\delta| < \beta - \alpha$, which gives

$$\frac{\beta^2}{4} |g_2(e^{i\varphi})|^2 = \beta(\beta - 2\alpha) x^2 + 2\delta(\beta - \alpha) x + \alpha^2 + \delta^2,$$

we obtain the results of Notaris [8] on the norm of the error functional, using his conditions (2.3_1)–(2.4_2) on the parameters $\alpha, \beta, \gamma, \delta$ under which the function $w(x, |g_2(e^{i\varphi})|^2)/w(-x, |g_2(e^{i\varphi})|^2)$ is strictly increasing, resp. strictly decreasing.

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